CLAUDIO BORRI
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LECTURES ON SOLID MECHANICS

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Contents

List of figures

Foreword

These Lecture Notes introduce the theoretical basics of solid mechanics to environmental engineering students. Born out of and supported by the European Project DEREC TEMPUS JEP Development of Environmental and Resources Engineering Curriculum, it collects the lectures held by the Authors during the course of Mechanic of Solids at the University of Florence, Degree of Environmental Engineering and Resources. Although the course is extended to basic structural engineering principles, such as mechanics, statics, kinematics and fundamental equations of beam structures, inertia, iso static and hyper static solution methods, these Lecture Notes reflect only the content of the lectures of continuum mechanics.

Several approaches are possible to the subject depending on the concern, either mathematically or physically oriented. The volume aims to provide a synthesis of both approaches, presenting in an organic whole the classical theory of solid mechanics and a more direct engineering approach. It is the Authors' opinion that a top–down learning process may offer to the engineering students those critical and autonomy tools necessary to gain awareness of that continuous learning process that is required; it characterizes the cultural and technical personality of an engineer. An ongoing learning is all the more necessary today, where the rapid development of powerful computers and computer solving methods (finite element methods, discrete volume methods, boundary methods, etc.) have opened up the way to new horizons that the classical approaches were only able to formulate. This fast and impressive growth of computer methods seems to be replacing the importance of gaining a consolidated knowledge of solid mechanics background. On the contrary, the Authors believe that only a conscious knowledge of theory can be that cultural instrument through which an engineer can really hope to control the use of computer methods. With this aim, the Reader addressed by this volume is mainly the undergraduate student in Engineering Schools: it is organized in eight Chapters: Chapter 1 proposes a synthesis of the basic concepts of mathematics and geometry that the readers need in the following chapters. Chapter 2 and Chapter 3 are devoted to the elementary framework of strain and stress in an elastic body. The concept of finite strain and Cauchy stress state is introduced, together with Mohr's representation of a general state of stress. Chapter 4 focuses on the classical law of linear elasticity. Chapter 5 deals with the Principle of Virtual Works. Chapter 6 treats the energy principles and provides a basic introduction to the variational methods.

Finally, Part I ends with a chapter introducing the notion of strength of materials. At the end of each chapter of the first part the basics of the tensor–based shell theory are also presented and then an application to some standard shell geometries is provided in appendix A.

The second part, Chapter 8, is dedicated to De Saint–Venant's problem where the classical beam theory is presented focusing on the four fundamental cases: beam under axial forces, terminal couples, torsion, bending and shear.

The volume, that consolidates the Lecture Notes prepared by the Authors for the second–year undergraduate students in environmental engineering, proposes a widening of the classical theories approached, giving a list of references used during its preparation as a possible suggestion to the Reader.

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Claudio Borri, Michele Betti, Enzo Marino

PART I

Theory of elasticity

Chapter 1
Outline of linear algebra

This chapter briefly presents some preliminary mathematics necessary to understand continuum mechanics. To this end the basic concepts of linear algebra and tensor analysis will be introduced. At the end of the chapter an overview of the theory of surfaces will be exposed in order to make the reader familiar with some background required for the mechanics of shell continuums, even though the latter is not the key theme of this book.

This introduction is neither exhaustive nor complete; indeed for any further insight the reader is warmly recommended to refer to the main sources from which this summary has been derived: Modugno, *[4] and [5];* Sokolnikoff, *[1];* Green-Zerna, *[3].*

1.1 Vector spaces and linear mappings

1.1.1 Vector spaces

We define *vector space* a set $\bar{V}$ equipped with the following operations

$$+ : \bar{V} \times \bar{V} : (\bar{u}, \bar{v}) \mapsto \bar{u} + \bar{v} \tag{1.1}$$

$$\cdot : I\!R \times \bar{V} : (\lambda, \bar{v}) \mapsto \lambda \bar{v}. \tag{1.2}$$

Elements belonging to $\bar{V}$ are named *vectors* and are characterized by the following properties

1. $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w} \qquad \forall\, \bar{u}, \bar{v}, \bar{w} \in \bar{V}$

2. $\bar{u} + \bar{v} = \bar{v} + \bar{u} \qquad \forall\, \bar{u}, \bar{v} \in \bar{V}$

3. $\bar{u} + \bar{0} = \bar{u} \qquad \forall\, \bar{u} \in \bar{V}$

4. $\forall\, \bar{u} \in \bar{V}\ \exists = -\bar{u} \in \bar{V}$ so that $\bar{u} + (-\bar{u}) = \bar{0}$

where $\bar{0}$ is called null vector.

Every vector space admits the existence of a subset

$$\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_n\} \subset \bar{V}$$

called the *basis* of $\bar{V}$. Thus, each vector $\bar{v} \in \bar{V}$ can be univocally represented through the basis $\mathcal{B}$ as follows

$$\bar{v} = v^i \bar{b}_i \qquad i = 1, \ldots, n \tag{1.3}$$

where $v^i \in I\!R$ are the components of $\bar{v}$ related to the basis $\mathcal{B}$ and n is a number which defines the dimension of $\bar{V}$, namely the number of vectors in any basis of $\bar{V}$.

Notice that in equation (1.3) the Einstein's summation convention has been used. It is a notational convenience where any term in which an index appears twice will stand for the sum of all such terms as the index assumes all of a preassigned range of values, hence

$$\bar{v} = v^i \bar{b}_i = \sum_{i=1}^{n} v^i \bar{b}_i \tag{1.4}$$

1.1.2 Linear mappings

Functions between two vector spaces assume a crucial importance in linear algebra. In particular, we define a linear map as a linear transformation between two vector spaces that preserves the operations of vector addition and scalar multiplication.

Let $\bar{V}$ and $\bar{V}'$ be two vector spaces equipped with the bases

$$\mathcal{B} = \{\bar{b}_1, \ldots, \bar{b}_n\}, \quad \mathcal{B}' = \{\bar{b}'_1, \ldots, \bar{b}'_m\}$$

respectively.

We define a *linear mapping* as the transformation

$$f : \bar{V} \to \bar{V}', \quad \bar{v} \mapsto \bar{v}' \tag{1.5}$$

if the two following conditions are satisfied

1. $f(\bar{u} + \bar{v}) = f(\bar{u}) + f(\bar{v}) \qquad \forall\, \bar{u}, \bar{v} \in \bar{V}$: additivity;

2. $f(\lambda \bar{u}) = \lambda f(\bar{u}) \qquad \forall\, \bar{u} \in \bar{V}$ e $\lambda \in I\!R$: homogeneity.

The set of all linear maps from $\bar{V}$ to $\bar{V}'$, denoted by $L\left(\bar{V},\bar{V}'\right)$, represents a $n\times m$–dimensional vector space, where n and m are the dimensions of $\bar{V}$ and $\bar{V}'$, respectively.

$$\{f:\bar{V}\rightarrow\bar{V}'\}=:L\left(\bar{V},\bar{V}'\right) \tag{1.6}$$

For linear mappings the following properties hold

1. $(f+g)(\bar{u})=f(\bar{u})+g(\bar{u})\,,\quad\forall f,g\in L\left(\bar{V},\bar{V}'\right);\,\bar{u}\in\bar{V}$
2. $(\lambda f)(\bar{u})=\lambda f(\bar{u})\,,\quad\forall f\in L\left(\bar{V},\bar{V}'\right);\,\bar{u}\in\bar{V}$

Matrix representation

Notions so far introduced allow us to assert that if f is a linear mapping from $\bar{V}$ to $\bar{V}'$, then $f(\bar{v})$ is a vector in $\bar{V}'$. Consequently, by recalling the expression in components for $\bar{v}$, (1.3), we have

$$f(\bar{v})=f(\bar{v})^i\,\bar{b}'_i\quad i=1,\ldots,m \tag{1.7}$$

and accounting for the fact that $\bar{v}=v^j\bar{b}_j$, with $j=1,\ldots,n$, and by using the homogeneity property for linear mappings, the latter equation leads to

$$f\left(v^j\bar{b}_j\right)^i\bar{b}'_i=v^jf\left(\bar{b}_j\right)^i\bar{b}'_i\quad j=1,\ldots,n\quad i=1,\ldots,m. \tag{1.8}$$

In a shorter form the components of $f(\bar{v})$ are then

$$(f(\bar{v}))^i=f^i_jv^j \tag{1.9}$$

so that the $m\times n$–dimensional matrix $f^i_j=f\left(\bar{b}_j\right)^i$ is the matrix representation of the linear mapping f referred to the bases $\mathcal{B}$ e $\mathcal{B}'$.

Linear forms and the dual space

Linear forms are a special case of linear mappings. Let $\bar{V}$ be a vector space and $\mathcal{B}=\{\bar{b}_i\}$ its basis. A *linear form* $\underset{\sim}{\omega}$ is a linear transformation from $\bar{V}$ to a scalar field

$$\underset{\sim}{\omega}:\bar{V}\rightarrow I\!R \tag{1.10}$$

Hence, we define $\bar{V}^*$ as the set of linear forms from $\bar{V}$ to $I\!R$

$$\bar{V}^*=:\{\omega:\bar{V}\rightarrow I\!R\}=:L\left(\bar{V},I\!R\right) \tag{1.11}$$

$\bar{V}^*$ and $\bar{V}$ have the same dimension.

The dual space $\bar{V}^*$ admits a basis $\mathcal{B}^* = \{\underline{\beta}^i\}$ whose elements are linear forms operating as follows

$$\underline{\beta}^i\left(\bar{b}_j\right) = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.12}$$

By the definition, we can state that the element $\underline{\beta}^i$ belonging to $\mathcal{B}^*$, applied to the vector $\bar{u}$, yields a scalar that is the i–th component of $\bar{u}$. In fact we write

$$\underline{\beta}^i\left(\bar{u}\right) = \underline{\beta}^i\left(u^j \bar{b}_j\right) = u^j \underline{\beta}^i\left(\bar{b}_j\right) = u^j \delta^i_j = u^i \tag{1.13}$$

We highlight that, as done for vectors, each linear form, chosen the n–dimensional basis $\mathcal{B}^*$, can be written in components as follows

$$\underline{\omega} = \omega_j \underline{\beta}^j \quad j = 1, \ldots, n \tag{1.14}$$

Bilinear forms

We can define a *bilinear form* $\underline{f}$ as a mapping

$$\underline{f} : \bar{V} \times \bar{V} \to I\!R, \quad \left(\bar{v}, \bar{v}'\right) \mapsto \lambda \tag{1.15}$$

where $\bar{v}, \bar{v}' \in \bar{V}$ and $\lambda \in I\!R$, and such that it is linear in each argument separately. That is

1. $\underline{f}\left(\bar{v} + \bar{w}, \bar{v}'\right) = \underline{f}\left(\bar{v}, \bar{v}'\right) + \underline{f}\left(\bar{w}, \bar{v}'\right)$;
2. $\underline{f}\left(\bar{v}, \bar{v}' + \bar{w}\right) = \underline{f}\left(\bar{v}, \bar{v}'\right) + \underline{f}\left(\bar{v}', \bar{w}\right)$;
3. $\underline{f}\left(\lambda \bar{v}, \bar{v}\right) = \underline{f}\left(\bar{v}, \lambda \bar{v}\right) = \lambda \underline{f}\left(\bar{v}, \bar{v}'\right)$.

$\forall f \in L\left(\bar{V} \times \bar{V}, I\!R\right); \ \bar{v}, \bar{v}', \bar{w} \in \bar{V}; \ \lambda \in I\!R.$

Endomorphisms

Frequently in the field of solid mechanics we will meet special linear mappings from a vector space into itself, i.e. $f \in L\left(\bar{V}, \bar{V}\right)$. These are defined *endomorphisms*

$$f : \bar{V} \to \bar{V}, \quad \bar{v} \mapsto \bar{v}' \qquad \bar{v}, \bar{v}' \in \bar{V} \tag{1.16}$$

The set of linear mappings from $\bar{V}$ into itself forms a $n \times n$–dimensional vector space, where n is the dimension of $\bar{V}$.

$$\{f : \bar{V} \to \bar{V}\} =: L\left(\bar{V}, \bar{V}\right) =: End\left(\bar{V}\right) \tag{1.17}$$

Change of basis for endomorphisms

Let $\mathcal{B}$ be a fixed basis for $\bar{V}$, we are interested in evaluating how the endomorphism $f \in End\left(\bar{V}\right)$ changes when passing to a new basis $\mathcal{B}'$ of $\bar{V}$. The following transformation rules are established

$$\bar{b}_i = a_i'^h \bar{b}'_h \tag{1.18}$$
$$\bar{b}'_h = a_h^j \bar{b}_j \tag{1.19}$$

that, by replacing (1.19) into (1.18), yield

$$\bar{b}_i = a_i'^h a_h^k \bar{b}_k \tag{1.20}$$

and so

$$\left(a_i'^h a_h^k - \delta_i^k\right) \bar{b}_k = 0 \Rightarrow a_i'^h a_h^k = \delta_i^k \tag{1.21}$$

therefore, each change of basis for $\bar{V}$ is characterized by a square invertible matrix $n \times n$.

Likewise vectors, the following rules hold for dual elements

$$\underline{\beta}^i = a_h^i \underline{\beta}'^h \tag{1.22}$$
$$\underline{\beta}'^i = a_h'^i \underline{\beta}^h \tag{1.23}$$

When both bases are orthogonal, then the transformation matrices are also orthogonal, that is

$$a_i'^h = a_h^i \tag{1.24}$$

where $a_i'^h = \left(a_i^h\right)^{-1}$, and

$$a_j'^i = \cos\left(\bar{b}'_i, \bar{b}_j\right) \tag{1.25}$$
$$a_k^h = \cos\left(\bar{b}_h, \bar{b}'_k\right) \tag{1.26}$$

The change of basis implies a change of the vector components. In fact we have

$$v^k = a_j^k v'^j \tag{1.27}$$
$$v'^k = a_j'^k v^j \tag{1.28}$$

The proof of the above equations can be easily provided. For instance, for equation (1.27) we have that a vector $\bar{v}$ can be expressed with respect to two basis $\mathcal{B}$ and $\mathcal{B}'$ as $\bar{v} = v^i \bar{b}_i = v'^j \bar{b}'_j$. Hence

$$v^i \bar{b}_i = v'^j a^k_j \bar{b}_k \Rightarrow v'^j a^k_j \bar{b}_k - v^i \bar{b}_i = 0 \Rightarrow \tag{1.29}$$

$$v'^j a^k_j \bar{b}_k - v^i \delta^k_i \bar{b}_k = 0 \Rightarrow \left(v'^j a^k_j - v^i \delta^k_i \right) \bar{b}_k = 0 \tag{1.30}$$

finally, by putting zero the coefficient in brackets, we obtain relation (1.27).

Covector components change by the the following rules

$$v_k = a'^i_k v'_i \tag{1.31}$$

$$v'_k = a^i_k v_i \tag{1.32}$$

Furthermore, recalling equation (1.9), via some manipulations, we get the rule to transform the endomorphism f, that is[1]

$$f^i_j = a^i_h f'^h_k a'^k_j \tag{1.33}$$

and

$$f'^i_j = a'^i_h f^h_k a^k_j \tag{1.34}$$

Similar relationships can be found for higher order matrices, for instance for a mixed fourth–order tensor we have

$$f^{ij}_{hk} = a^i_l a^j_m f'^{lm}_{no} a'^n_h a'^o_k \tag{1.35}$$

and likewise

$$f'^{ij}_{hk} = a'^i_l a'^j_m f^{lm}_{no} a^n_h a^o_k \tag{1.36}$$

1.2 Euclidean spaces

A Euclidean vector space is a space which admits a Euclidean metric, that is a structure inducing some special relationships between distances and angles. In particular, fixed a Cartesian coordinate system (that will be better defined later on) and its standard basis, in a Euclidean space the distance between two points can be computed by means of *Pitagora*'s formula.

[1] Often, within an engineering context, it is convenient to represent equations (1.33) and (1.34) in the matrix form, such as $F' = R^T F R$ and $F = R F' R^T$, where R^T and R are nothing but a'^i_j and a^h_k, respectively.